Twisted magnetar magnetospheres: a class of semi-analytical force-free non-rotating solutions

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April 9, 2025

arXiv2504.00680 or hal-05014723 :
arxiv.org/abs/2504.00680 or hal.science/hal-05014723

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Introduction

Motivations

- Modelling of internal magnetar magnetospheres, observables, opacity;
- Twist (toroidal component) : effect of starquakes, relaxation and powering of radiation;
- Coupled evolution between stellar interior and magnetosphere:
  - Requires freedom in setting boundary conditions
- Production of Fast Radio Bursts by magnetar magnetospheres.

The problem

- In absence of rotation, the force-free hypothesis and axial symmetry leads to

$$\vec{\nabla} \times \vec{B} = \alpha(\mathcal{P})\vec{B}, \tag{1}$$

where α is a function such that $\vec{j} = \alpha\vec{B}$ is the current density, and $\mathcal{P} = \mathrm{constant}$ along field lines.

- Magnetic field:

$$\vec{B} = \vec{B}_{\mathrm{p}} + B_\varphi \vec{e}_\varphi \text{ with } \vec{B}_{\mathrm{p}} = \frac{\vec{\nabla}\mathcal{P} \times \vec{e}_\varphi}{r \sin\theta}, \tag{2}$$

Unknowns are α (or A) and $\mathcal{P}$

- From $\vec{\nabla} \times \vec{B} = \alpha \vec{B}$:

$$A = \int \alpha \mathrm{d}\mathcal{P} = r \sin\theta B_\varphi . \quad (3)$$

- The Grad-Shafranov equation

$$-\partial_r^2 \mathcal{P} - \frac{1-\mu^2}{r^2} \partial_\mu^2 \mathcal{P} = \alpha(\mathcal{P}) A(\mathcal{P}), \quad (4)$$

where $\mu \equiv \cos\theta$.

- Boundary conditions :

$$\mathcal{P}(\mu = \pm 1) = 0 \quad (5)$$

Example: self-similar solutions (Low& Lou 90, Wolfson 95, Thompson+ 02)

- Find an ansatz:

$$\mathcal{P} = F(\mu)/r^p \;\; ; \;\; \alpha(\mathcal{P}) = c\mathcal{P}^{1/p}, \tag{6}$$

with F a function to determine, c and p are constants.

- Grad-Shafranov separates:

$$-p(p+1)F - (1-\mu^2)\partial^2_\mu F = cF^{1+2/p}, \tag{7}$$

- Properties: $\vec{B} \propto r^{-(p+2)}$, non-linear.
- Fixing $F(\mu=\pm 1) = 0, \partial_\mu F(1), c, p$ the equation is overdetermined...

Example: numerical solutions

- $$\alpha = s(\mathcal{P} - \mathcal{P}_c)^\sigma \text{ for } \mathcal{P} > \mathcal{P}_c \qquad (8)$$

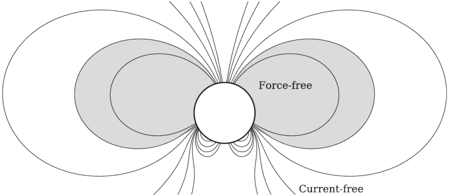


Figure: From Akgün+ 2016

- Time dependent force-free (e.g. Parfrey+13)

The solution

Ansatz

$$\mathcal{P} = \sum_{i=1} \frac{F_i(\mu)}{r^i}, \quad (9)$$

$$\alpha(\mathcal{P}) = \sum_{i=1} (i+1) c_i \mathcal{P}^i, \quad (10)$$

$$A(\mathcal{P}) = \sum_{i=1} c_i \mathcal{P}^{i+1}, \quad (11)$$

where $F_i(\mu)$ are functions to be determined, and c_i are a priori arbitrary constants.

Hierarchy

For each order $1/r^i$,

$$\forall i \geq 1, -i(i+1)F_i - (1-\mu^2)F_i'' = [\alpha A]_{l=i+2}. \qquad (12)$$

Complex source terms :

$$[\alpha A]_3 = 2c_1^2 G_3^{(3)}, \qquad (13)$$

$$[\alpha A]_4 = 2c_1^2 G_4^{(3)} + 5c_1 c_2 G_4^{(4)}, \qquad (14)$$

$$[\alpha A]_5 = 2c_1^2 G_5^{(3)} + 5c_1 c_2 G_5^{(4)} + (6c_1 c_3 + 3c_2^2) G_5^{(5)}, \qquad (15)$$

$$\ldots \qquad (16)$$

with

$$G_l^{(k)} = \sum_{i_1+\ldots+i_k=l; i_j \geq 1} F_{i_1} \ldots F_{i_k}. \qquad (17)$$

Properties

- If $c_i = 0$, each equation generates the corresponding vacuum multipole at order $1/r^i$ e.g. $F_1 \to$ dipole.
- If $|c_1| > 0$, then highly non-linear. For example at order $1/r$:

$$-2F_1 - (1-\mu^2)F_1'' = 2c_1^2 F_1^3. \tag{18}$$

- If $c_1 = 0$ then F_1 is a vacuum dipole and all equations are linear, but higher orders are sourced by it e.g. for $c_2 \neq 0$ at order $1/r^3$,

$$-6F_3 - (1-\mu^2)F_3'' = 3c_2^2 F_1^5. \tag{19}$$

- Generally, the equation for F_i is hierarchically sourced by $F_{j\leq i}$.

Properties

- Symmetries:
  - Parity: $\{F_i(\mu)\} \to \{F_i(-\mu)\}$
  - Twist orientation: $\{c_i\} \to \{-c_i\} \Rightarrow B_\varphi \to -B_\varphi$
- Convergence:
  - Series converges for $|c_i|$ and $|F_i'|$ not too large.
  - In the case $c_{i\neq 2} = 0, F_{i>1} = 0$, we find convergence for $|c_2| \lesssim 6F_1'^2$.

$$\mathcal{P} = \sum_{i=1} \frac{F_i}{r^i}, \quad A = \sum_{i=1} c_i \mathcal{P}^{i+1}, \quad B_\varphi = \frac{A}{r \sin\theta}$$
$$\forall i \geq 1, -i(i+1)F_i - (1-\mu^2)F_i'' = [\alpha A]_{l=i+2},$$

Equatorial current sheet

- In order to satisfy boundary conditions $F_i(\mu = \pm 1) = 0$ one can solve one hemisphere and symmetrise using even parity $\{F_i(\mu)\} \to \{F_i(-\mu)\}$. This generates a toroidal equatorial current sheet with density,

$$\sigma_\varphi = -2B_r(\mu = 0). \quad (20)$$

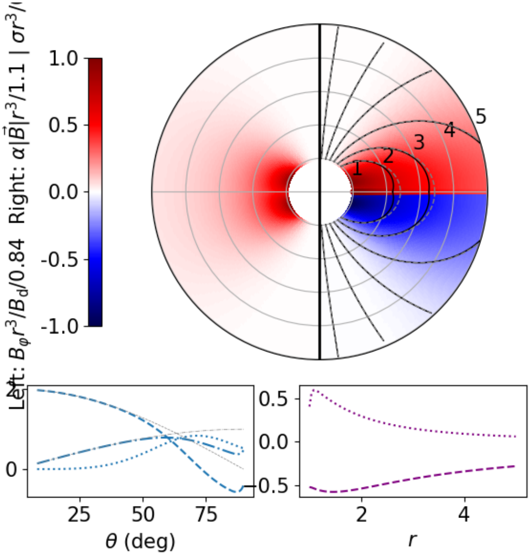
- Twist orientation can be reversed across equator $\{c_i\} \to \{-c_i\} \Rightarrow B_\varphi \to -B_\varphi$, yielding a radial current sheet,

$$\sigma_r = -2B_\varphi(\mu = 0). \quad (21)$$

- Supported by quadrupole and higher-multipole orders i.e. $1/r^{i \geq 4}$.

Examples: Picture time !

Example 1: Dipole with $c_2 = 6, F_1 = 1$



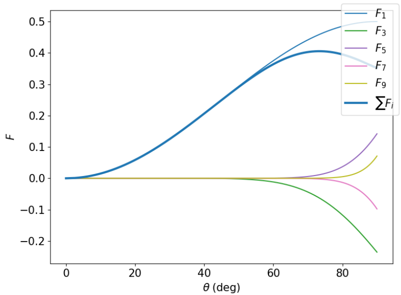


Figure: First 10 F_i.

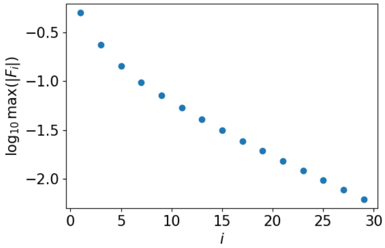


Figure: Convergence

- Leading r^{-4} correction: dipole F_1 and sourced twist,

$$\vec{B} = \vec{B}_1^{\mathrm{V}} \pm c_2 \frac{B_1^3}{6} \frac{(1-\mu^2)^{5/2}}{r^4} \vec{e}_\varphi + \bigcirc\left(\frac{1}{r^5}\right). \qquad (22)$$

- Matches radial dependence fitted to FRB data using a geometrical model (Voisin 2023; Voisin & Francez, Submitted, hal-05022825)

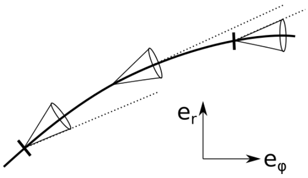
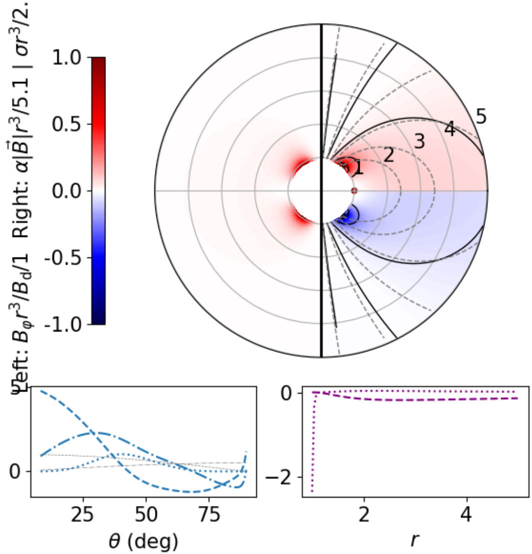


Figure: Emission cone along twisted field line (Voisin 2023)

- Analytical solution worked out up to octupolar order.

Example 2: Dipole+Octupole, $c_2 = 3.2, \dot{F}_1 = 1, \dot{F}_3 = 4$



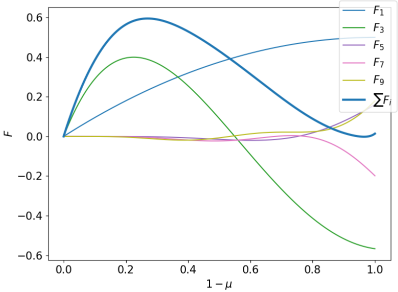


Figure: $\dot{F}_1 = 1, \dot{F}_3 = 4, c_2 = 3.2$. The thick line shows the sum of the first 50 orders.

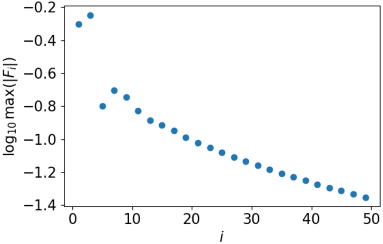
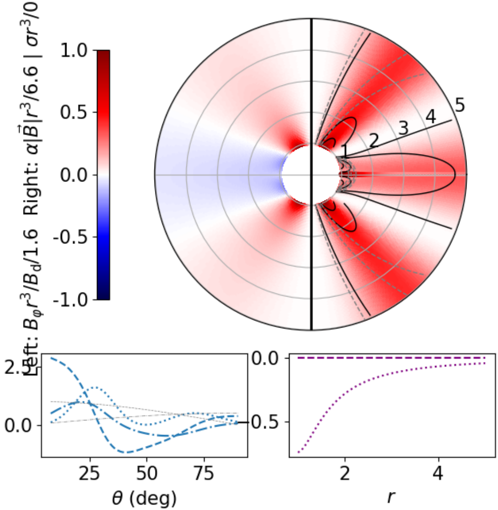


Figure: Convergence

Example 3: Non-Vacuum dipole+Octupole, $c_1 = 15.981364889, \dot{F}_1 = 1, \dot{F}_3 = 2$



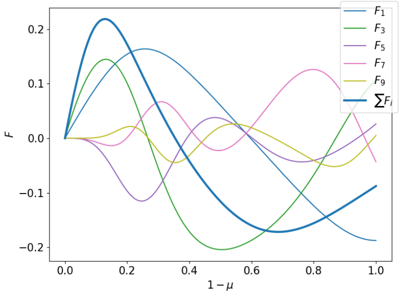


Figure: The thick line shows the sum of the first 70 orders.

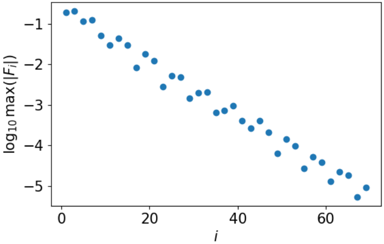


Figure: Convergence

Conclusions

Conclusions

- Multipolar expansion: great freedom in choosing boundary conditions
- Equatorial current sheet both toroidal, and radial if B_φ is reversed.
- Solutions asymptotically tending to vacuum.
- Simple analytical solution at quadrupolar order:
  - Matches $1/r^4$ dependence found in fitting geometrical model to FRB data (Voisin & Francez, Submitted, hal-05022825).

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Backup slides

Analytical octupolar contribution for $c_{i\neq 2} = 0$

In principle analytical solutions are possible e.g. the equation for F_3 giving the octupole contribution at r^{-5} is sourced by F_1 and has solution

$$F_3 = K_3\left(-\frac{8}{15}\bar{\mu}^6 + \frac{88}{105}\bar{\mu}^7 - \frac{24}{49}\bar{\mu}^8 + \frac{113}{882}\bar{\mu}^9 - \frac{17}{1323}\bar{\mu}^{10} - \frac{1}{97020}\bar{\mu}^{11} + \sum_{i=12}^{\infty}\alpha_i\bar{\mu}^i\right), \quad (23)$$

with K_3 a constant, and $\bar{\mu} = 1 - \mu$.

The problem

In absence of rotation, the force-free hypothesis leads to

$$\vec{\nabla} \times \vec{B} = \alpha \vec{B}, \qquad (24)$$

where α is a function such that $\vec{j} = \alpha \vec{B}$ is the current density. Taking the divergence of Eq. (24) + axial symmetry,

$$\vec{B} \cdot \vec{\nabla} \alpha = 0, \qquad (25)$$

such that α is constant along a field line.

$$\vec{B} = \vec{B}_{\rm p} + B_\varphi \vec{e}_\varphi \text{ with } \vec{B}_{\rm p} = \frac{\vec{\nabla}\mathcal{P} \times \vec{e}_\varphi}{r \sin\theta}, \qquad (26)$$

with $\mathcal{P} = \mathrm{constant}$ along field lines, and $\alpha = \alpha(\mathcal{P})$.